Chapter 2

Lines, Planes and Quadric Surfaces

Linear and quadratic functions are the simplest functions. These functions and their zero sets as subsets of the Euclidean space are studied. Ideas of analytic geometry sketched in Section 2.1 link these algebraic objects to geometry. Hyperplanes and straight lines associated to linear functions and systems of linear functions are discussed in Sections 2.2 and 2.3 respectively. Quadratic curves and quadric surfaces associated to quadratic functions are classified in Section 2.4.

2.1 Algebra and Geometry

Geometry, invented by Ancient Greeks, is a gem in human civilization. In plane geometry geometric figures such as triangles and circles are studied. All results, no matter how intricate and profound, depend purely on logical deduction from a few axioms. Such axiomatic approach to a branch of knowledge which solely depends on rigorous reasoning has inspired the development of not only science but also other fields. Newton's Mathematica Principia that laid the foundation of mechanics was written in the style of Euclid's Elements. On the other side, algebra was a different subject in mathematics. Originated from arithmetic, numbers were replaced by symbols in the operations. Equations are no longer solved one by one, instead there are general formulas which yield the desired results after plugging in the numbers.

Geometry and algebra remained as two separated branches of mathematics for more than two thousand years until the French scholar René Descartes (1596-1650) introduced the so-called Cartesian coordinates (or rectangular coordinates). Every point on the coordinate line corresponds to a real number and every point in the plane corresponds to an ordered pair. Using the coordinate system, the algebraic equation $f(x, y) = 0$, where f is a polynomial as long as algebra is concerned, is turned into a geometric object, namely, the set

$$\{(x, y) : \ f(x, y) = 0 \ \} \ ,$$

in the plane. In this way algebra and geometry are combined in an intimate way.

Given a function in n-many variables, we let

$$\Sigma = \{\mathbf{x} \in \mathbb{R}^n : \ f(\mathbf{x}) = 0\} \ ,$$

be its **zero set** of f. It is also called the **solution set** of f as one can also view $f(\mathbf{x}) = 0$ as solving an equation. When expressed in the form

$$\Sigma_c = \{\mathbf{x} \in \mathbb{R}^n : \ f(\mathbf{x}) = c\} \ ,$$

which corresponds to the zero set of the function $f - c$, this set is called the **level set** of the function f at c. The terminologies of a zero set, a solution set and a level set will be used frequently in these notes. Essentially they represent the same things viewed from different angle. The equation $f(\mathbf{x}) = 0$ falls in the category of algebra when f is a polynomial but the zero set is a geometric object. The simplest algebraic equation is the linear equation

$$\sum_{j=1}^{n} a_j x_j = b \ ,$$

where not all a_j's vanish. Its solutions form the zero set of the linear function

$$p(\mathbf{x}) = \sum_{j=1}^{n} a_j x_j - b \ .$$

The zero set is a straight line when $n = 2$ and a plane when $n = 3$. Next to the linear equations are the quadratic equations, whose general form is given by

$$\sum_{j,k=1}^{n} a_{jk} x_j x_k + \sum_{j=1}^{n} b_j x_j = c \ ,$$

where not all a_{jk}'s vanish. When $n = 2$, its zero sets include circles, ellipses, hyperbolas and parabolas. Straight lines and circles are the primary objects of study in plane geometry. After the introduction of analytic geometry we see that they are associated to linear and quadratic equations respectively. In the following sections we will study hyperplanes, straight lines and quadric hypersurfaces using this approach.

2.2 Hyperplanes

Consider the linear equation in $\mathbb{R}^n, n \geq 1$,

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b \ , \quad a_1, a_2, \cdots, a_n, b \in \mathbb{R} \ .$$

Here it is implicitly assumed at least one of the coefficients a_j's is non-zero. It is called a **homogeneous linear equation** when $b = 0$ and a **non-homogeneous linear equation** when $b \neq 0$. Using the dot product, we can write a linear equation in the form

$$\mathbf{a} \cdot \mathbf{x} = b \ , \quad \mathbf{a} \in \mathbb{R}^n \ , \ b \in \mathbb{R} \ .$$

Given a linear equation, its solution set

$$\{\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : \ a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \ \},$$

is called the **hyperplane** associated to the equation. When $n = 2$, the hyperplane is called the **straight line** or simply the line associated to the equation. When $n = 3$, it is called the **plane** associated to the equation. Our definition here illustrates the idea of Descartes that an algebraic concept has a geometric name. In fact, we may define the plane in a purely geometric way. First consider the plane passing through the origin. Roughly speaking, it should consist of all points (position vectors) that are perpendicular to fixed non-zero vector pointing in the normal direction of the plane. In symbols, letting $\mathbf{a} = (a_1, a_2, a_3)$ be the **normal vector**, the plane is given by the set

$$\left\{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : \ \mathbf{a} \cdot \mathbf{x} = 0\right\} ,$$

which coincides with our definition of a plane associated to the homogeneous equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$. The normal vector is not unique; when $\mathbf{a}$ is a normal vector, any non-zero multiple of it is also a normal vector. For a plane not passing through the origin, we translate the origin to any fixed point on the plane. Thus, letting the point be $\mathbf{p}$, the plane should consist of all points satisfying

$$\left\{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : \quad \mathbf{a} \cdot (\mathbf{x} - \mathbf{p}) = 0\right\} ,$$

which is the same as

$$\left\{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : \quad \mathbf{a} \cdot \mathbf{x} = b\right\} , \quad b = \sum_j a_j p_j \ .$$

We have shown that a plane not passing through the origin is associated to a non-homogeneous linear equation. The same reasoning applies to $n = 2$ which shows that the geometric notion of a straight line coincides with the algebraic definition given above. When $n \geq 4$, figures cannot be drawn but the idea is still valid. A hyperplane passing through the origin is associated a homogeneous linear equation and the hyperplane associated to a non-homogeneous equation not passing through the origin.

How to write down the equation of a hyperplane? The principle is very simple: A hyperplane is completely determined when its normal vector and a point on it are known. In other words, letting $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ be a vector that is perpendicular to the plane and $\mathbf{p}$ a point on the plane, the equation of the hyperplane is given by

$$\mathbf{a} \cdot (\mathbf{x} - \mathbf{p}) = 0 \ ,$$

or

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b, \quad b = \mathbf{a} \cdot \mathbf{p} \ .$$

Example 2.1. Find the equation of the plane which is parallel to the plane $2x-7y+z=0$ and passing through the point $(1, 2, 3)$. By parallel we mean these two planes have the same normal direction. Therefore, the sought-after plane has normal $(2, -7, 1)$ and, as it passes through $(1, 2, 3)$, $b = (2, -7, 1) \cdot (1, 2, 3) = -9$. The equation for the plane is given by

$$(2, -7, 1) \cdot ((x, y, z) - (1, 2, 3)) = 0 \ ,$$

that is, $2x - 7y + z = -9$.

Example 2.2. Find the straight line passing through $(-1, 2)$ and is perpendicular to the straight line $2x + 5y = -9$. The normal direction of the line $2x + 5y = -9$ is $(2, 5)$, so the normal direction of our straight line is $(5, -2)$ (you may choose $(-5, 2)$ as well). Therefore, the equation of our straight line is

$$(-5, 2) \cdot ((x, y) - (-1, 2)) = 0, \quad \text{or } -5x + 2y = 9 \ .$$

Sometimes we are asked to find the equation of the hyperplane passing certain points. For $n = 3$, it is apparent three points determine a plane uniquely unless they are collinear. There points $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbb{R}^3$ are collinear if $\alpha\mathbf{p} + \beta\mathbf{q} + \gamma\mathbf{r} = \mathbf{0}$ for some non-zero numbers α, β, γ. In higher dimension, it is more tedious to describe the conditions that n points determined a hyperplane in geometric terms. However, using linear algebra, we see that n points determine a hyperplane uniquely provided they are linearly independent.

Given linearly independent $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \in \mathbb{R}^n$, in order to determine the equation of the hyperplane passing through these points it suffices to determine its normal direction, say $\mathbf{a}$, which should satisfy the linear system

$$(\mathbf{p}_1 - \mathbf{p}_n) \cdot \mathbf{a} = 0 \ , \quad (\mathbf{p}_2 - \mathbf{p}_n) \cdot \mathbf{a} = 0 \ , \cdots , \quad (\mathbf{p}_{n-1} - \mathbf{p}_n) \cdot \mathbf{a} = 0 \ .$$

This is a system of n unknowns and $n - 1$ equations. As the number of equations is less than the number of unknowns, it is always solvable. When the $n - 1$ points $\mathbf{p}_j - \mathbf{p}_n, j = 1, \cdots, n - 1$, are linearly independent, it is known that the solution is one dimensional, that is, it is spanned by a single vector, and we can take it to be the normal. The problem of determining the hyperplane is reduced to solving a linear system.

When $n = 3$, we can take advantage of the cross product. Now we need to solve $(\mathbf{p}_1 - \mathbf{p}_3) \cdot \mathbf{a} = 0$ and $(\mathbf{p}_2 - \mathbf{p}_3) \cdot \mathbf{a} = 0$. Recalling $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$, we see that a normal direction is given by $(\mathbf{p}_1 - \mathbf{p}_3) \times (\mathbf{p}_2 - \mathbf{p}_3)$.

Theorem 2.1. *The equation for the plane passing three non-collinear points $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ is given by*

$$\Big((\boldsymbol{p}_1 - \boldsymbol{p}_3) \times (\boldsymbol{p}_2 - \boldsymbol{p}_3)\Big) \cdot \big((x, y, z) - \boldsymbol{p}_3\big) = 0 \ .$$

Example 2.3. Find the equation of the plane passing through the points

$$(0, 1, 1),\ (2, 3, 0),\ (2, 3, 4) \ .$$

We compute

$$((0, 1, 1) - (2, 3, 4)) \times ((2, 3, 0) - (2, 3, 4)) = (-2, -2, -3) \times (0, 0, -4) = (8, -8, 0) \ ,$$

so the equation is given by

$$(8, -8, 0) \cdot ((x, y, z) - (2, 3, 4)) = 8(x - 2) - 8(y - 3) + 0(z - 4) = 0,$$

that is, $x - y + 1 = 0$. We are free to choose which point to be subtracted from, for instance, now we take $(0, 1, 1)$ to replace $(2, 3, 4)$. Then

$$((2, 3, 0) - (0, 1, 1)) \times ((2, 3, 4) - (0, 1, 1)) = (2, 2 - 1) \times (2, 2, 3) = (8, -8, 0) \ ,$$

so the equation is

$$(8, -8, 0) \cdot ((x, y, z) - (2, 3, 0)) = 8(x - 2) - 8(y - 3) - 0z = 0 \ ,$$

which yields the same equation $x - y + 1 = 0$.

Example 2.4. Find the equation of the plane passing through the points

$$(1, 1, 1),\ (2, -1, 0),\ (0, -3, 4) \ .$$

Although the cross product approach may be used, let us follow the general approach. First, we bring $(1, 1, 1)$ to the origin.

$$(2, -1, 0) - (1, 1, 1) = (1, -2, -1), \quad (0, -3, 4) - (1, 1, 1) = (-1, -4, 3) \ .$$

The normal direction of the plane (a, b, c) is perpendicular to these two vectors,

$$(1, -2, -1) \cdot (a, b, c) = 0, \quad (-1, -4, 3) \cdot (a, b, c) = 0 \ ,$$

which gives

$$a - 2b - c = 0, \quad a + 4b - 3c = 0 \ .$$

Using c as the parameter, we solve this system to get $a = 5c/3$ and $b = c/3$. That is, the vector $(5/3, 1/3, 1)c$ is perpendicular to the plane. Taking $c = 3$, our plane satisfies the equation

$$(5, 1, 3) \cdot ((x, y, z) - (1, 1, 1)) = 0, \qquad \text{i.e., } 5x + y + 3z = 9 \ .$$

Next we derive a formula for the distance from a point to a hyperplane.

Theorem 2.2. *Let $\boldsymbol{p} \in \mathbb{R}^n$ and $\boldsymbol{a} \cdot \boldsymbol{x} = b$ be a hyperplane in $\mathbb{R}^n$. The distance from $\boldsymbol{p}$ to the hyperplane is given by*

$$\frac{|\boldsymbol{a} \cdot \boldsymbol{p} - b|}{|\boldsymbol{a}|} .$$

Proof. Our derivation of the formula is based on the observation that the distance is equal to the length of the line segment from $\mathbf{p}$ perpendicular to the hyperplane. Let $\mathbf{q}$ be the point on the plane so that $\mathbf{p} - \mathbf{q}$ is perpendicular to the hyperplane. We have two equations, namely,

$$\mathbf{a} \cdot \mathbf{q} = b \ , \quad \mathbf{p} - \mathbf{q} = \lambda \mathbf{a} \ , \quad \lambda \in \mathbb{R} \ .$$

The first equation means $\mathbf{q}$ is a point on the plane and the second equation means $\mathbf{p} - \mathbf{q}$ points to the normal direction of the hyperplane. We plug $\mathbf{q} = \mathbf{p} - \lambda \mathbf{a}$ in the first equation to get

$$\mathbf{a} \cdot (\mathbf{p} - \lambda \mathbf{a}) = b \ ,$$

which yields

$$\lambda = \frac{\mathbf{a} \cdot \mathbf{p} - b}{|\mathbf{a}|^2} \ .$$

It follows that

$$\mathbf{q} = \mathbf{p} - \lambda \mathbf{a} = \mathbf{p} - \frac{\mathbf{a} \cdot \mathbf{p} - b}{|\mathbf{a}|^2} \mathbf{a} \ .$$

As the distance from $\mathbf{p}$ to the hyperplane is given by $|\mathbf{p} - \mathbf{q}|$, we conclude that it is given by

$$\frac{|\mathbf{a} \cdot \mathbf{p} - b|}{|\mathbf{a}|} \ .$$

□

Corollary 2.3. *Let $\boldsymbol{a} \cdot \boldsymbol{x} = b$ and $\boldsymbol{a} \cdot \boldsymbol{x} = c$ be two parallel planes. The distance between them is given by $|b - c|/|\boldsymbol{a}|$.*

Here distance between two parallel planes is the minimal distance between the points from different planes.

Proof. The distance from a point $\mathbf{p}$ on the second plane to the first one is given by $|\mathbf{a} \cdot \mathbf{p} - b|/|\mathbf{a}|$, and the formula follows after noting $\mathbf{a} \cdot \mathbf{p} = c$. □

Let $H^+ : \mathbf{a} \cdot \mathbf{x} > b$ and $H^- : \mathbf{a} \cdot \mathbf{x} < b$. Then the origin belongs to H^- when $b > 0$ and to H^+ when $b < 0$. The normal vector $\mathbf{a}$ points from H^- to H^+. Plugging $\mathbf{p} = \mathbf{0}$ in this formula, we see that the distance from the origin to H is given by $|b|$ when $\mathbf{a}$ is a unit vector. This gives a meaning to b.

Example 2.5. Let $H: \ 2x + 5y - z + w = 2$ be a hyperplane in $\mathbb{R}^4$ and $P(1, 2, 0, -3)$ a point lying outside the hyperplane.

(a) Find the point on the hyperplane that realizes the distance between P and H.

(b) Find the distance from P to H.

We apply the formulas:

$$\mathbf{a} \cdot \mathbf{p} = (2, 5, -1, 1) \cdot (1, 2, 0, -3) = 9,$$

and

$$\lambda = \frac{\mathbf{a} \cdot \mathbf{p} - b}{|a|^2} = \frac{9-2}{31} = \frac{7}{31} \ ,$$

hence

$$\mathbf{q} = \mathbf{p} - \lambda \mathbf{a} = (1, 2, 0, -3) - \frac{7}{31}(2, 5, -1, 1) = \frac{1}{31}(17, 27, 7, -100) \ ,$$

which is the answer to (a) and

$$|\mathbf{p} - \mathbf{q}| = \frac{7}{\sqrt{31}} \ ,$$

is the answer to (b).

We introduce one more terminology. Let $H: \ \mathbf{a} \cdot \mathbf{x} = 0$ be a hyperplane passing through the origin and some $\mathbf{p} \in \mathbb{R}^n$. The **projection** of $\mathbf{p}$ on the hyperplane H is the point $\mathbf{q}$ sitting on H so that $\mathbf{p} - \mathbf{q}$ is perpendicular to H, that is,

$$(\mathbf{p} - \mathbf{q}) \cdot \mathbf{a} = 0 \ , \quad \mathbf{a} \cdot \mathbf{q} = 0.$$

From the proof above we see that the projection of $\mathbf{p}$ on H is given by

$$\mathbf{q} = \mathbf{p} - \frac{\mathbf{a} \cdot \mathbf{p}}{|\mathbf{a}|^2} \mathbf{a} \ .$$

2.3 Straight Lines

In the spirit of analytic geometry, one would like to define straight lines in terms of linear equations. A moment's reflection shows that any possible definition would depend on the dimension. In $n = 2$, a straight line is the solution set of a single linear equation as seen in the last section. However, for $n = 3$, it arises as the intersection of two planes, that is, it is the solution set of the system of two linear equations

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1, \\ a_2 x + b_2 y + c_2 z = d_2 \ . \end{cases}$$

In fact, to avoid these two planes being parallel so the solution set becomes empty, it has to postulate that these two planes are linearly independent (more precisely, they do not have the same normal direction). In general, for $n \geq 2$, we may define a straight line to be the solution set of $n-1$-many linearly independent hyperplanes. However, such approach is a little bit indirect and a more direct one like the parametric representation discussed below is preferred.

As a straight line is determined by its direction and a point it passes through. Here we will adapt a definition that is motivated by kinetics where s straight lines is viewed as the trajectory of a particle moving along the same direction in constant speed. To be specific, given $\mathbf{p} \in \mathbb{R}^n$ and a non-zero $\boldsymbol{\xi} \in \mathbb{R}^n$, a **straight line** passing through $\mathbf{p}$ along the direction determined by $\boldsymbol{\xi}$ is given by the set of points

$$\{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{x} = \mathbf{p} + \boldsymbol{\xi} t \ , \quad t \in \mathbb{R}\} \ .$$

As t could be any real number, $\boldsymbol{\xi}$ in the definition would be fine as long as it is non-zero. However, it would degenerate into the single point $\{\mathbf{p}\}$ when $\boldsymbol{\xi}$ is the zero vector. Hence $\boldsymbol{\xi}$ cannot be the zero vector in the definition. Also, it is not necessary to be a unit vector. For the set is the same as

$$\left\{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{x} = \mathbf{p} + \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\, t \ , \quad t \in \mathbb{R}\right\} .$$

From a kinetic point of view, the particle moves from $\mathbf{p}$ at $t = 0$ along the line in the direction of $\boldsymbol{\xi}/|\boldsymbol{\xi}|$. In some old texts, the equation of a straight line is expressed as

$$\frac{x_1 - p_1}{\xi_1} = \frac{x_2 - p_2}{\xi_2} = \cdots = \frac{x_n - p_n}{\xi_n} \ ,$$

which is an alternate description that the straight line passing $\mathbf{p}$ with slope $\boldsymbol{\xi}$. In fact, letting t be the common ratio, this expression can be converted to $\mathbf{x} = \mathbf{p} + t\boldsymbol{\xi}$.

Now we show the two approaches are equivalent.

Theorem 2.4. *Any straight line is the intersection of two linearly independent planes in $\mathbb{R}^3$. Conversely, the solution set of two linear equations with different normal directions is a straight line for some $\boldsymbol{p}$ and $\boldsymbol{\xi}$.*

Proof. * Given a straight line $\mathbf{p} + t\boldsymbol{\xi}$, we can find two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$ satisfying $\mathbf{u} \cdot \boldsymbol{\xi} = 0, \mathbf{v} \cdot \boldsymbol{\xi} = 0$. The plane passing through $\mathbf{p}$ with normal $\mathbf{u}$ is given by the equation $\mathbf{u} \cdot (\mathbf{x} - \mathbf{p}) = 0$. Similarly, the plane passing through $\mathbf{p}$ with normal $\mathbf{v}$ is given by $\mathbf{v} \cdot (\mathbf{x} - \mathbf{p}) = 0$. The solution set of these two equation consists of all points $\mathbf{x}$ satisfying $\mathbf{u} \cdot (\mathbf{x} - \mathbf{p}) = 0, \ \mathbf{v} \cdot (\mathbf{x} - \mathbf{p}) = 0$ simultaneously. Points on the straight line are of the form $\mathbf{x} = \mathbf{p} + t\boldsymbol{\xi}$. By $\mathbf{u} \cdot (\mathbf{x} - \mathbf{p}) = \mathbf{u} \cdot t\boldsymbol{\xi} = t\mathbf{u} \cdot \boldsymbol{\xi} = 0$ and $\mathbf{v} \cdot (\mathbf{x} - \mathbf{p}) = \mathbf{v} \cdot t\boldsymbol{\xi} = t\mathbf{v} \cdot \boldsymbol{\xi} = 0$

we see that the straight line is contained in the solution set. On the other hand, letting $\mathbf{x}$ be a point in the solution set, the conditions $\mathbf{u}\cdot(\mathbf{x}-\mathbf{p})=0,\ \mathbf{v}\cdot(\mathbf{x}-\mathbf{p})=0$ mean that $\mathbf{x}-\mathbf{p}$ is perpendicular to $\mathbf{u}$ and $\mathbf{v}$, hence it must point to the direction of $\boldsymbol{\xi}$. Thus, there is some t_1 such that $\mathbf{x}-\mathbf{p}=t_1\boldsymbol{\xi}$, i.e., $\mathbf{x}=\mathbf{p}+t_1\boldsymbol{\xi}$, $\mathbf{x}$ is a point on the straight line. We have shown that the straight line and the solution set coincide.

Next, consider the linear system

$$\begin{cases} a_1x+b_1y+c_1z=d_1, \\ a_2x+b_2y+c_2z=d_2\ , \end{cases}$$

where (a_1,b_1,c_1) and (a_2,b_2,c_2) are linearly independent. The matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$$

has rank 2. From linear algebra there must be at least one non-singular 2×2-submatrix. Assuming it is from the first two columns and rows, we move the z-terms to the right and write the system as

$$\begin{cases} a_1x+b_1y=-c_1z+d_1, \\ a_2x+b_2y=-c_2z+d_2\ . \end{cases}$$

Solve this system to get $x=\alpha+\beta z$ and $y=\gamma+\delta z$ for some $\alpha,\beta,\gamma,\delta$. Thus the solution set consists of $(\alpha,\gamma,0)+(\beta,\delta,1)z,\ z\in\mathbb{R}$, that is, it is a straight line.

□

Example 2.6. Find the expression for the straight lines which is the intersection of the planes

$$\begin{cases} x+y+z=1, \\ 2x-y+6z=5,\ . \end{cases}$$

We may take z as the "time parameter" and write the system as

$$\begin{cases} x+y=1-z, \\ 2x-y=5-6z. \end{cases}$$

Solve this equation to get

$$x=\frac{1}{3}(6-7z)\ ,\quad y=\frac{1}{3}(-3+4z)\ .$$

Writing $t=z$, the straight line is given by

$$(x,y,z)=\left(\frac{1}{3}(6-7t),\frac{1}{3}(-3+4t),t\right)=(2,-1,0)+\left(-\frac{7}{3},\frac{4}{3},1\right)t\ ,t\in\mathbb{R}\ .$$

It passes through $(2,-1,0)$ at $t=0$ with constant velocity $(-7/3,4/3,1)$.

Alternatively, we can take y as the time parameter. We write

$$\begin{cases} x + z = -y + 1, \\ 2x + 6z = 5 + y, \end{cases}$$

which gives

$$x = \frac{1}{4}(1 - 7y) \ , \quad z = \frac{1}{4}(3 + 3y) \ ,$$

so the straight line can be described as

$$(x, y, z) = \left(\frac{1}{4}, 0, \frac{3}{4}\right) + \left(-\frac{7}{4}, 1, \frac{3}{4}\right) t \ , \quad t \in \mathbb{R} \ .$$

Observing

$$\left(-\frac{7}{4}, 1, \frac{3}{4}\right) = \frac{3}{4}\left(-\frac{7}{3}, \frac{4}{3}, 1\right) \ ,$$

we see that they represent the same set. Only now the particle starts at $(1/4, 0, 3/4)$ with constant velocity $(-7/4, 1, 3/4)$. Although in these two formulas the motions are different, the geometry is the same.

It is not hard to see that either x, y or z can be chosen to be the time parameter as long as the 2×2-matrix obtained after moving the chosen variable to the other side is non-singular.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The straight line passing through $\mathbf{x}$ and $\mathbf{y}$ is given by $\mathbf{x} + t(\mathbf{y} - \mathbf{x}) = (1 - t)\mathbf{x} + t\mathbf{y}, \ t \in \mathbb{R}$. When $t = 0$, it gives the point $\mathbf{x}$ and, when $t = 1$, it is $\mathbf{y}$. You could image this is the path of a particle moving from $\mathbf{x}$ to $\mathbf{y}$ in constant speed so that it arrives at $\mathbf{y}$ at a unit time. Likewise we can use the expression $\mathbf{y} + t(\mathbf{x} - \mathbf{y})$ which the particle now moves from $\mathbf{y}$ to $\mathbf{x}$. In particular, we see the line segment between $\mathbf{x}$ and $\mathbf{y}$ corresponds to the time interval $[0, 1]$. This way of describing a line segment is very useful as we will see.

Example 2.7. Consider the triangle whose vertices are $A(0, 0), \ B(2, 0), \ C(1, 1)$. Find

(a) Its medium from A,

(b) Its height from A,

(c) * Its bisector from A.

(a) The midpoint of the side $\overline{BC}$ is given by $((2, 0) + (1, 1))/2 = (3, 1)/2$. The vector $(3, 1)$ points to the direction of the median. As the median passes $A(0, 0)$, the median is given by the set

$$\left\{(3, 1)t : \quad t \in \left[0, \frac{1}{2}\right]\right\} \ .$$

(b) Let $\overline{AD}$ be the height from A where D is on the side BC. Let D be $(2,0)+t((1,1)-(2,0)) = (2-t,t)$ where t is to be specified. The direction of AD is perpendicular to BC whose direction points in $(1,1)-(2,0)=(-1,1)$. Noting $\overrightarrow{AD}\perp\overrightarrow{BC}$, we have

$$(-1,1)\cdot(2-t,t)=0\ ,$$

which is readily solved to get $t=1$. We conclude that $D=C$ and the height coincides with AC. In other words, this is a perpendicular triangle with the right angle at C.

(c)* Let $\theta=\angle CAB$. The lengths of AB and AC are given by 2 and $\sqrt{2}$ respectively. By the Cosine Law,

$$\cos\theta=\frac{(2,0)\cdot(1,1)}{2\sqrt{2}}=\frac{\sqrt{2}}{2}\ .$$

Using the half angle formula,

$$\cos\frac{\theta}{2}=\left(\frac{1+\cos\theta}{2}\right)^{1/2}=a\ ,\quad a=\frac{\sqrt{2+\sqrt{2}}}{2}\ .$$

The direction of the bisector is given by

$$\left(\cos\frac{\theta}{2},\sin\frac{\theta}{2}\right)=(a,b),\quad b=\sqrt{1-a^2}=\frac{\sqrt{2-\sqrt{2}}}{2}\ .$$

On the other hand, BC is given by $(2,0)+((1,1)-(2,0))s=(2-s,s),\ s\in[0,1]$. The line $(0,0)+t(a,b)$ hits $\overline{BC}$ at $t(a,b)=(2-s,s)$. Solving for t and s, we get $t=2/(a+b)$ and $s=2b/(a+b)$. We conclude that the bisector at A is given by

$$\left\{(a,b)t:\quad t\in\left[0,\frac{2}{a+b}\right]\right\}\ .$$

2.4 Quadric Hypersurfaces

A quadric hypersurface is defined as the zero set or the solution set Σ of a quadratic equation

$$\sum_{j,k=1}^{n}a_{jk}x_jx_k+\sum_{j=1}^{n}b_jx_j+c=0,$$

where not all a_{jk}'s are zero. Let us start with quadratic curves, that is, $n=2$. We write the equation as

$$ax^2+2bxy+cy^2+dx+ey=f\ , \tag{2.1}$$

and denote its solution set by γ. Alternatively we can express the equation in the form

$$(x,y)\begin{bmatrix}a & b\\ b & c\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+dx+ey=f\ .$$

To study the geometry of γ we simplify the equation by rotating the coordinates which do not alter the shape of γ. The following theorem will be used to classify the curves defined by a quadratic equation.

Theorem 2.5. *For any symmetric* 2×2*-matrix*

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} ,$$

there is a rotation

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

such that

$$R'AR = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} ,$$

where λ *and* μ *are eigenvalues of the symmetric matrix. Consequently, letting*

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} ,$$

we have

$$ax^2 + 2bxy + cy^2 = \lambda u^2 + \mu v^2 .$$

Proof.

$$\begin{aligned} R'AR &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} a\cos^2\theta + 2b\sin\theta\cos\theta + c\sin^2\theta & b\cos 2\theta + \dfrac{c-a}{2}\sin 2\theta \\ b\cos 2\theta + \dfrac{c-a}{2}\sin 2\theta & a\sin^2\theta + c\cos^2\theta - b\sin 2\theta \end{bmatrix} . \end{aligned}$$

We can always choose some $\theta_0 \in [0, \pi)$ such that

$$b\cos 2\theta_0 + \frac{c-a}{2}\sin 2\theta_0 = 0,$$

so that $R'AR = D$ where D is a diagonal matrix

$$D = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} .$$

From

$$R'AR\, \mathbf{e}_1 = \lambda \mathbf{e}_1, \quad R'AR\, \mathbf{e}_2 = \mu \mathbf{e}_2 ,$$

we see that

$$A\mathbf{x} = \lambda\mathbf{x}, \quad A\mathbf{y} = \mu\mathbf{y} \ ,$$

where

$$\mathbf{x} = R\mathbf{e}_1, \quad \mathbf{y} = R\mathbf{e}_2 \ .$$

It shows that λ and μ are in fact the eigenvalues of A. $\square$

By introducing the new variables u, v as described in this theorem, our quadratic equation turns into another quadratic equation

$$\lambda u^2 + \mu v^2 + du + ev = f \ , \tag{2.2}$$

for different d and e. Since the shape of γ remains unchanged under rotations, it suffices to study γ by assuming the equation is (2.2).

Theorem 2.6. *Consider equation (2.1).*

(1) If λ and μ are of the same sign, there is a Euclidean motion under which the equation assumes the form

$$|\lambda|x^2 + |\mu|y^2 = c \ , \quad c \in \mathbb{R}.$$

Consequently, γ is either an ellipse ($c > 0$), a point ($c = 0$) or an empty set ($c < 0$).

(2) If λ and μ are of different sign, there is a Euclidean motion under which the equation assumes the form

$$|\lambda|x^2 - |\mu|y^2 = c \ , \quad c \in \mathbb{R} \ .$$

Consequently, γ is either a hyperbola ($c \neq 0$), or the union of two intersecting straight lines ($c = 0$).

(3) If one of λ, μ is zero, there is a Euclidean motion under which the equation assumes the form

$$|\lambda|x^2 + ay = c \ , \quad a, c \in \mathbb{R} \ .$$

Consequently, γ is either a parabola ($a \neq 0$), two parallel straight lines ($a = 0, c > 0$), the empty set ($a = 0, c < 0$) or a straight line ($a = c = 0$).

Proof. By the previous theorem, we may assume that the equation is already in the form (2.2).

If λ and μ are of the same sign. By multiplying -1 to this equation if necessary, we may assume they are positive. By completing square, it becomes

$$\lambda\left(x + \frac{d}{2\lambda}\right)^2 + \mu\left(y + \frac{e}{2\mu}\right)^2 = g \ , \quad g = f + \frac{d^2}{2\lambda} + \frac{e^2}{4\mu} \ .$$

Therefore, after a translation

$$u = x + \frac{d}{2\lambda} , \quad v = y + \frac{e}{2\mu} ,$$

we achieve at $\lambda u^2 + \mu v^2 = g$. When $g > 0$, this is the standard form for an ellipse. When $g = 0$, it degenerates into a single point. When $g < 0$, this equation has no solution, so γ is an empty set.

If λ and μ are of opposite sign. By multiplying -1 to this equation if necessary, we may assume λ is positive and μ is negative. Following the discussion in the first case, we arrive at $|\lambda|u^2 - |\mu|v^2 = g$. When $g \neq 0$, γ is a hyperbola. When $g = 0$, it is the union of the straight lines defined by

$$\sqrt{|\lambda|}u + \sqrt{|\mu|}v = 0 , \quad \sqrt{|\lambda|}u - \sqrt{|\mu|}v = 0 .$$

If one of λ, μ is zero, by switching the x- and y-axis if necessary, we may assume $\lambda > 0$ and $\mu = 0$ so that the equation becomes

$$\lambda x^2 + dx + ey + f = 0 ,$$

for some new f. A partial completing square yields

$$\lambda \left(x + \frac{d}{2\lambda} \right)^2 + ey + f - \frac{d^2}{4\lambda} = 0 .$$

Hence, after a horizontal translation, the equation becomes $\lambda x^2 + ey = f$. It is a parabola as long as $e \neq 0$. When $e = 0$ and $f > 0$, γ consists of two vertical lines $x = \pm\sqrt{f}$. It is empty when $e = 0$ and $f < 0$. It is the y-axis when $e = f = 0$.

We have completely classified the curves defined by quadratic equations of two variables. □

Remark 2.1. The switching of coordinates is realized by the linear transformation $x \mapsto y, y \mapsto x$. It is obtained by a rotation of 90° followed by a reflection of the y-axis. It is again a Euclidean motion.

Remark 2.2. λ and μ are of the same sign iff $ac - b^2 > 0$. They are of opposite sign iff $ac - b^2 < 0$. One of λ, μ vanishes iff $ac - b^2 = 0$. This follows from the relation

$$ac - b^2 = \det A = \lambda\mu .$$

Steps of transforming a quadratic equation into the "standard form" are:

Step 1. Solve the characteristic equation

$$\det \begin{bmatrix} a-\lambda & b \\ b & c-\lambda \end{bmatrix} = 0 \ ,$$

to determine the two eigenvalues λ_1 and λ_2 including multiplicity.

Step 2. Solve the linear systems

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$$

and

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$$

to obtain two orthogonal unit eigenvectors (u_1, v_1) and (u_2, v_2).

Step 3. The change of variables

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

will convert the equation in x, y into one in u, v without mixed term uv.

Step 4. Completing square to bring it into the standard form.

Example 2.8. Transform the equation

$$2xy - x + 3y = 1$$

to the standard form and determine its solution set. We have $a = c = 0$ and $b = 1$ so $ac - b^2 = -1 < 0$ and there are two eigenvalues with opposite sign. In fact, the characteristic polynomial is $\lambda^2 - 1 = 0$ so the two eigenvalues are 1 and -1 with corresponding eigenvector $(1, 1)$ and $(-1, 1)$ so that

$$R = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \ .$$

This is the rotation by $45°$. Note that the factor $\sqrt{2}/2$ is for normalization. Letting

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \ ,$$

that is,

$$x = \frac{\sqrt{2}}{2}(u - v), \quad y = \frac{\sqrt{2}}{2}(u + v) \ ,$$

$$\begin{aligned}
2xy - x + 3y - 1 &= u^2 - v^2 - \frac{\sqrt{2}}{2}(u-v) + \frac{3\sqrt{2}}{2}(u+v) - 1 \\
&= u^2 - v^2 + \sqrt{2}u + 2\sqrt{2}v - 1 \\
&= (u + \frac{\sqrt{2}}{2})^2 - (v - \sqrt{2})^2 + \frac{1}{2} \ ,
\end{aligned}$$

where in the last step we complete square. Letting $x' = u + \sqrt{2}$ and $y' = v + \sqrt{2}$, the equation finally achieves the standard form $x^2 - y^2 = -\frac{1}{2}$ which is a hyperbola after replacing (x', y') by (x, y).

In fact, since the key step is to get rid of the mixed term xy, a short cut is: we simply set

$$x = au + bv, \quad y = cu + dv \ ,$$

in the equation to get

$$2(au + bv)(cu + dv) - (au + cv) + 3(cu + dv) - 1 = 0 \ .$$

The mixed term is given by $4(ad + bc)$ which can be killed off by choosing $a = d = b = 1$ and $c = -1$. The resulting equation becomes $-2u^2 + 2v^2 - (u - v) + 3(-u + v) - 1$, that is, $-2(u + 1)^2 + 2(v + 1)^2 - 1 = 0$. Setting $x = v + 1$ and $y = u + 1$, we arrive at

$$x^2 - y^2 - \frac{1}{2} = 0 \ .$$

The hyperbola obtained in this way is similar but not congruent to the hyperbola described by the original equation. But it serves the purpose if one just wants to know the shape of the zero set of the equation. Henceforth, any equation without mixed terms and lower order terms can be called a standard form of the equation.

The situation for all other dimensions is similar, thanks to the following basic result in linear algebra: For any symmetry matrix A, there is an orthogonal matrix R such that $R'AR = D$ where D is a diagonal matric whose diagonal elements are precisely the eigenvalues of A (counting multiplicity), see the Comments below. Using this result, a suitable Euclidean motion would bring the general quadratic equation into

$$\sum_{j=1}^{n} \lambda_j x_j^2 + \sum_{j=1}^{n} b_j x_j + c = 0 \ ,$$

and further classification according to the sign of the eigenvalues can be carried out as in the two variable case. Here let us state what happens in $n = 3$.

First of all, under a Euclidean motion the equation is of the form

$$\lambda x^2 + \mu y^2 + \nu z^2 + dx + ey + fz = g \ . \tag{2.3}$$

We have

Theorem 2.7. *Consider equation (2.3).*

(a) If λ, μ, ν are of the same sign, there is a Euclidean motion to transform (2.3) to

$$|\lambda|x^2 + |\mu|y^2 + |\nu|z^2 = g\ , \quad g \in \mathbb{R}\ . \quad (\textbf{\textit{ellipsoid}})$$

(b) If two of λ, μ, ν are of the same sign and one in opposite sign, there is a Euclidean motion to transform (2.3) to

$$|\lambda|x^2 + |\mu|y^2 - |\nu|z^2 = g\ , \quad g \in \mathbb{R}\ .$$

*(**hyperboloid of one sheet** $g > 0$, **elliptical cone** $g = 0$, **hyperboloid of two sheets** $g < 0$)*

(c) If two of λ, μ, ν are of the same sign and the third one is zero, there is a Euclidean motion to transform (2.3) to

$$|\lambda|x^2 + |\mu|y^2 + fz = g\ , \quad f, g \in \mathbb{R}\ . \quad (\textbf{\textit{elliptical paraboloid}})$$

(d) If one of λ, μ, ν is zero and the other two are in opposite sign, there is a Euclidean motion to transform (2.3) to

$$|\lambda|x^2 - |\mu|y^2 + fz = g\ , \quad f, g \in \mathbb{R}\ . \quad (\textbf{\textit{hyperbolic paraboloid}})$$

(e) If exactly two of λ, μ, ν are zero, there is a Euclidean motion to transform (2.3) to

$$|\lambda|x^2 + ey = g\ , \quad e, g \in \mathbb{R}\ .$$

This theorem can be established by following the same reasoning as the two dimensional case. We point out that in (e) one would get $\lambda x^2 + ey + fz = g$, but then a rotation of the yz-plane cancels the z-term.

We have studied planes, straight lines, quadratic curves and quadric surfaces regarding them as the zero sets of a single equation or a system of linear equations. It is natural to investigate what geometric objects one would obtain as the zero sets for more complicated equations or systems. We will return to this question in Chapter 6 after we have equipped with the knowledge of differentiation theory.

Comments on Chapter 2

2.1 The standard form of quadratic curves are:

$$\begin{aligned}
&\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ , \quad a, b > 0 \text{ (ellipse)}\ ,\\
&\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\ , \quad a, b > 0 \text{ (hyperbola)}\ ,\\
&x = 2py^2\ , \quad p \in \mathbb{R}\ , \text{(parabola)}\ .
\end{aligned}$$

We will say more about these curves in Chapter 3.

2.2. We will use frequently a theorem from linear algebra concerning the diagonalization of a symmetric matrix. To state first recall that an orthogonal matrix is an $n \times n$-matrix whose columns are unit vectors orthogonal to each other, that is, $R = (r_{ij})$ satisfies for each i, k, $\sum_j^n r_{ji} r_{jk} = \delta_{jk}$. In terms of the matrix product, letting R' be the transpose matrix of R, an orthogonal matrix satisfies $R'R = RR' = Id$.

Principal Axis Theorem. *Let $A = (a_{ij})$ be a symmetric matrix. There exists an orthogonal matrix R such that $R'AR = D$ where D is diagonal.*

In Section 7.5 a proof of this theorem based on calculus will be given. The main step in the proof is to establish the following fact: There are $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ such that

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} u_1^k \\ \cdot \\ \cdot \\ u_n^k \end{bmatrix} = \lambda_k \begin{bmatrix} u_1^k \\ \cdot \\ \cdot \\ u_n^k \end{bmatrix} ,$$

for some non-zero vector $\mathbf{u}^k$. Each $\mathbf{u}^k$ is a unit eigenvector associated to the eigenvalue λ_k. Eigenvectors to different eigenvalues are perpendicular to each other and those associated to the same eigenvalue can be chosen to be perpendicular. In this way the matrix

$$R = \begin{bmatrix} u_1^1 & \cdots & u_1^n \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ u_n^1 & \cdots & u_n^n \end{bmatrix} .$$

is orthogonal and satisfies the requirement in the theorem.

This theorem can be used to simplify the quadratic function

$$q(x) = \sum_{i,j=1}^n a_{ij} x_i x_j .$$

In fact, by replacing both a_{ij} and a_{ji} by $(a_{ij} + a_{ji})/2$, we may assume $A = (a_{ij})$ is a symmetric matrix. Consequently there is an orthogonal matrix $R = (r_{ij})$ and a diagonal matrix $D = (\lambda_k \delta_{jk})$ so that $R'AR = D$, that is,

$$\sum_{k,m} r'_{ik} a_{km} r_{mj} = \lambda_j \delta_{ij}.$$

Therefore, by the change of variables $\mathbf{x} = R\mathbf{y}$, we have

$$\begin{aligned} q(x) &= \sum_{i,j} a_{ij} x_i x_j \\ &= \sum_{i,j} a_{ij} \sum_{k,m} r_{ik} y_k r_{jm} y_m \\ &= \sum_{k,m} \sum_{i,j} r'_{ki} a_{ij} r_{jm} y_k y_m \\ &= \sum_{k,m} \lambda_k \delta_{km} y_k y_m \\ &= \sum_k \lambda_k y_k^2 \ . \end{aligned}$$

The quadratic function assumes a very simple form in the new variable $\mathbf{y}$.

Supplementary Reading

1.3 and 1.4 in [Au].